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Spin j Dirac operators on the fuzzy 2-sphere

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Abstract: The spin $\frac{1}{2}$ Dirac operator and its chirality operator on the fuzzy 2-sphere S_F^2 can be constructed using the Ginsparg-Wilson(GW) algebra [1]. This construction actually exists for any spin j on S_F^2, and have continuum analogues as well on the commutative sphere S^2 or on $\mathbb{R}^2$. This is a remarkable fact and has no known analogue in higher dimensional Minkowski spaces. We study such operators on S_F^2 and the commutative S^2 and formulate criteria for the existence of the limit from the former to the latter. This singles out certain fuzzy versions of these operators as the preferred Dirac operators. We then study the spin 1 Dirac operator of this preferred type and its chirality on the fuzzy 2-sphere and formulate its instanton sectors and their index theory. The method to generalize this analysis to any spin j is also studied in detail.

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1 Introduction

The Dirac and chirality operators are central for fundamental physics and also in non-commutative geometry, where it is used to formulate metrical, differential geometric and bundle-theoretic ideas following Connes' approach [2].

The theory of these operators on the fuzzy sphere S_F^2 can be formulated using the Ginsparg-Wilson(GW) algebra, or the approach of [3, 4]. The GW algebra was originally encountered in the context of lattice gauge theories [5] where it was formulated in order to avoid the fermion doubling problem. The fact that this algebra appears naturally in the fuzzy case is interesting. In particular we shall see that it provides a way to formulate the Dirac and chirality operators for any non-zero spin. The latter in turn leads to a Dirac-like equation for any spin on S^2 and $\mathbb{R}^2$ with its associated chirality operator.

We shall hereafter refer to these Dirac-like equations and their chiralities just as Dirac and chirality operators.

These Dirac and chirality operators remind one of the Duffin-Kemmer, Rarita-Schwinger and Bargmann-Wigner equations. The relation between these well-known equations and those found in this paper remain to be explored.

In section 2, we establish our notation for S_F^2 and recall the earlier formulation of the GW algebra and the fuzzy Dirac and chirality operators for spin $\frac{1}{2}$. In section 3, we examine the ambiguities in the construction of the fuzzy spin $\frac{1}{2}$ Dirac operator and study their continuum limits as well. When these limits exist, the resultant continuum operators are unitarily equivalent.

Section 4 gives a procedure to construct these operators in the continuum. These will act as a guide in taking the limit of their corresponding fuzzy versions, thereby fixing the fuzzy Dirac and chirality operators. With this in mind, we explicitly construct the spin 1 Dirac and chirality operators in the continuum.

Then in section 5 we go on to construct the fuzzy versions of the spin 1 Dirac operator by the construction of their GW algebras in such a manner that the continuum limits exist.

In section 6, guided by the analysis in section 5, the GW systems and hence their Dirac and chirality operators are constructed for any spin in such a manner that their continuum limits exist. Crucial observations of terms arising in such computations are made and the procedure for taking their continuum limit is discussed in detail.

In section 7 we summarize our rules for finding the fuzzy Dirac and chirality operators. We also prove a claim which unambiguously fixes the fuzzy Dirac and chirality operators for all spins.

Instanton sectors can be formulated in the algebraic language in terms of projective modules [6]. There is a natural adaptation of this idea to S_F^2 for scalar and spin $\frac{1}{2}$ fields [1, 7]. The index theory has also been established in the latter case. In section 8, we generalize this construction to any spin and their Dirac and chirality operators on S_F^2 and also establish their index theory.

In section 9 we present our conclusions.

2 The fuzzy sphere and its GW algebra

The algebra for the fuzzy sphere is characterized by a cut-off angular momentum L and is the full matrix algebra $Mat(2L+1) \equiv M_{2L+1}$ of $(2L+1) \times (2L+1)$ matrices. They can be generated by the $(2L+1)$-dimensional irreducible representation (IRR) of SU(2) with the standard angular momentum basis. The latter is represented by the angular momenta L_i^L acting on the left on $Mat(2L+1)$: If $\alpha \in Mat(2L+1)$,

$$L_i^L \alpha = L_i \alpha \tag{2.1}$$

$$[L_i^L, L_j^L] = i\epsilon_{ijk} L_k^L \tag{2.2}$$

$$(L_i^L)^2 = L(L+1)\mathbf{1} \tag{2.3}$$

where L_i are the standard angular momentum matrices for angular momentum L.

We can also define right angular momenta L_i^R:

$$L_i^R \alpha = \alpha L_i, \alpha \in M_{2L+1} \tag{2.4}$$

$$[L_i^R, L_j^R] = -i\epsilon_{ijk} L_k^R \tag{2.5}$$

$$(L_i^R)^2 = L(L+1)\mathbf{1}\,. \tag{2.6}$$

We also have

$$[L_i^L, L_j^R] = 0. \tag{2.7}$$

The operator $\mathcal{L}_i = L_i^L - L_i^R$ is the fuzzy version of orbital angular momentum. They satisfy the SU(2) angular momentum algebra

$$[\mathcal{L}_i, \mathcal{L}_j] = i\epsilon_{ijk}\mathcal{L}_k\,. \tag{2.8}$$

In the continuum, S^2 can be described by the unit vector $\hat{x} \in S^2$, where $\hat{x}.\hat{x} = 1$. Its analogue on S_F^2 is $\frac{L_i^L}{L}$ or $\frac{L_I^R}{L}$ such that

$$\lim_{L\to\infty} \frac{L_i^{L,R}}{L} = \hat{x}_i. \tag{2.9}$$

This shows that $L_i^{L,R}$ do not have continuum limits. But $\mathcal{L}_i = L_i^L - L_i^R$ does and becomes the orbital angular momentum as $L \to \infty$:

$$\lim_{L\to\infty} L_i^L - L_i^R = -i(\overrightarrow{r} \wedge \overrightarrow{\nabla})_i. \tag{2.10}$$

The GW algebra. In algebraic terms, the GW algebra $\mathcal{A}$ is the unital $*$ algebra over $\mathbf{C}$, generated by two $*$-invariant involutions Γ, Γ'.

$$\mathcal{A} = \{\Gamma, \Gamma' \;: \Gamma^2 = \Gamma'^2 = 1\,, \Gamma^* = \Gamma\,, \Gamma'^* = \Gamma'\} \tag{2.11}$$

In any $*$ -representation on a Hilbert space, $*$ becomes the adjoint $\dagger$.

Each representation of eq. (2.11) is a particular realization of the GW algebra. Representations of interest in fuzzy physics are generally reducible.

The Dirac operator from GW algebra. Consider the following two elements constructed out of Γ, Γ':

$$\Gamma_1 = \frac{1}{2}(\Gamma + \Gamma'), \tag{2.12}$$

$$\Gamma_2 = \frac{1}{2}(\Gamma - \Gamma'). \tag{2.13}$$

It follows from eq. (2.11) that $\{\Gamma_1, \Gamma_2\} = 0$. This suggests that for suitable choices of Γ, Γ', one of these operators may serve as the Dirac operator and the other as the chirality operator provided they have the right continuum limits after suitable scaling. This is indeed the case as we now show for the fuzzy spin $\frac{1}{2}$ Dirac and chirality operators.

The fuzzy Dirac operator: Spin $\frac{1}{2}$. The construction is based on the GW algebra of [8, 9]. First we note that if P is a projector, then,

$$P^2 = P \tag{2.14}$$

and $\gamma = 2P - 1$ is an idempotent:

$$\gamma^2 = 1. \tag{2.15}$$

We now construct Γ, Γ' from suitable projectors.

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Consider $Mat(2L+1)\otimes\mathbb{C}^2$. The spin $\frac{1}{2}$ IRR of SU(2) acts on $\mathbb{C}^2$. It has the standard Lie algebra basis $\frac{\sigma_i}{2}$, σ_i being the Pauli matrices. The projector coupling the left angular momentum and this spin $\frac{1}{2}$ to its maximum value $L+\frac{1}{2}$ is

$$P^L_{L+\frac{1}{2}} = \frac{\vec{\sigma}.\vec{L}^L + L + 1}{2L+1}. \tag{2.16}$$

Hence the corresponding idempotent is

$$\Gamma^L_{L+\frac{1}{2}} = \frac{\vec{\sigma}.\vec{L}^L + \frac{1}{2}}{L+\frac{1}{2}}. \tag{2.17}$$

The projector $P^R_{L+\frac{1}{2}}$ coupling the right angular momentum and spin $\frac{1}{2}$ to $L+\frac{1}{2}$ is obtained by changing $\vec{L}^L$ to $-\vec{L}^R$ in the above expression:

$$P^R_{L+\frac{1}{2}} = \frac{-\vec{\sigma}.\vec{L}^R + L + 1}{2L+1}. \tag{2.18}$$

The minus sign is because of the minus sign in eq. (2.5).

The corresponding idempotent is

$$\Gamma^R_{L+\frac{1}{2}} = \frac{-\vec{\sigma}.\vec{L}^R + \frac{1}{2}}{L+\frac{1}{2}}. \tag{2.19}$$

Identifying $\Gamma^{L,R}_{L+\frac{1}{2}}$ with Γ, Γ', we get

$$\Gamma_1 = \frac{1}{2}\left[\frac{\vec{\sigma}.\vec{\mathcal{L}}+1}{L+\frac{1}{2}}\right] \tag{2.20}$$

and

$$\Gamma_2 = \frac{1}{2}\left[\frac{\vec{\sigma}.(\vec{L}^L+\vec{L}^R)}{L+\frac{1}{2}}\right]. \tag{2.21}$$

Now as $L\to\infty$,

$$2L\Gamma_1 \to \vec{\sigma}.\vec{\mathcal{L}}+1 \tag{2.22}$$

and

$$\Gamma_2 \to \vec{\sigma}.\hat{x}. \tag{2.23}$$

These are the correct Dirac and chirality operators on S^2 and so we can regard $2L\Gamma_1$ as the fuzzy Dirac operator (upto a finite scaling) and Γ_2 as its chirality operator.

3 Ambiguities in the fuzzy spin $\frac{1}{2}$ Dirac and Chirality operators

Having looked at the construction of the spin $\frac{1}{2}$ Dirac operator as given in [1], we now consider other possibilities for constructing the same Dirac operator. This observation turns out to be crucial in finding the Dirac operator for higher spins.

The projectors $P^{L,R}_{L+\frac{1}{2}}$ are not the only projectors with rotational invariance. We can also consider the two projectors to the $L-\frac{1}{2}$ space, obtained by coupling the left and right angular momenta $L_i^{L,R}$ and spin $\frac{1}{2}$. These are,

$$P^L_{L-\frac{1}{2}} = -\left(\frac{\vec{\sigma}.\vec{L}^L - L}{2L+1}\right), \tag{3.1}$$

and

$$P^R_{L-\frac{1}{2}} = -\left(\frac{-\vec{\sigma}.\vec{L}^R - L}{2L+1}\right). \tag{3.2}$$

This gives us two new generators, $\Gamma^{L,R}_{L-\frac{1}{2}}$, to the GW algebra. Thus there are a total of four rotationally invariant idempotents which we list in the following table

$$P^{L,R}_{L+\frac{1}{2}}: \quad \Gamma^L_{L+\frac{1}{2}} \quad \Gamma^R_{L+\frac{1}{2}} \tag{3.3}$$

$$P^{L,R}_{L-\frac{1}{2}}: \quad \Gamma^L_{L-\frac{1}{2}} \quad \Gamma^R_{L-\frac{1}{2}} \tag{3.4}$$

The negatives of these idempotents are also idempotents, but that is a trivial ambiguity.

Now a GW algebra is generated by any pair from this table. However if we adopt the two left or the two right as Γ and Γ', then Γ_1 and Γ_2 have no suitable continuum limit. We can see this from choosing as our generators either $\Gamma^L_{L\pm\frac{1}{2}}$ or $\Gamma^R_{L\pm\frac{1}{2}}$. We observe that $\Gamma^L_{L+\frac{1}{2}} = -\Gamma^L_{L-\frac{1}{2}}$ and $\Gamma^R_{L+\frac{1}{2}} = -\Gamma^R_{L-\frac{1}{2}}$, which as remarked above is a trivial ambiguity. So clearly we cannot construct suitable GW algebras from such pairs of idempotents.

But if we now use the two operators $\Gamma^L_{L+\frac{1}{2}}$ and $\Gamma^R_{L-\frac{1}{2}}$ and consider the combination $(L+\frac{1}{2})(\Gamma^L_{L+\frac{1}{2}} - \Gamma^R_{L-\frac{1}{2}})$, we get the Dirac operator given in eq. (2.22). As we saw earlier in section 2 [1], this Dirac operator is found by adding $\Gamma^L_{L+\frac{1}{2}}$ and $\Gamma^R_{L+\frac{1}{2}}$ and scaling as $L \to \infty$. The corresponding chirality operator is got from $\frac{\Gamma^L_{L+\frac{1}{2}}+\Gamma^R_{L-\frac{1}{2}}}{2}$ as this goes to the correct limit as $L \to \infty$ which is $\sigma.\hat{x}$. The other possibility of combining $\Gamma^L_{L-\frac{1}{2}}$ and $\Gamma^R_{L+\frac{1}{2}}$ also exists and it is easy to see that $-(L+\frac{1}{2})(\Gamma^L_{L-\frac{1}{2}} - \Gamma^R_{L+\frac{1}{2}})$ also goes to the Dirac operator given by eq. (2.22) while $\frac{\Gamma^L_{L-\frac{1}{2}}+\Gamma^R_{L+\frac{1}{2}}}{2}$ goes to the corresponding chirality operator. This exhausts all the possible combinations.

We again note here that we can only construct our desired Dirac and chirality operators by choosing one Γ from the second column and one from the third column of eq. (3.3) and eq. (3.4) as we will not get a differential operator in the continuum if we choose them from the same column.

The fact that there exist all these possibilities for combining various generators of the GW algebra for obtaining the fuzzy Dirac and chirality operators imply that we should take care while writing the corresponding versions of higher spin Dirac and chirality operators as not all of them may go to correct continuum limits. In the case of spin $\frac{1}{2}$, all the possibilities go to the correct continuum limit, but as we shall soon see, this fails in the case of higher spins. This calls for a rule to construct the fuzzy versions of these operators, which we shall formulate after studying the spin 1 case in detail. We shall also see later that this becomes essential for finding the Dirac operators in the continuum for higher spins.

But there *are* more substantial ambiguities to consider. There are other operators in the GW algebra which can serve as Dirac and chirality operators [1, 9]. For example there are those which give the Dirac and chirality operators of the Watamuras [10] on S_2^F. As shown in [9], in the continuum limit, the corresponding operators are unitarily equivalent to eq. (2.22). We will not pursue such ambiguities further here.

In the next section we will see how to construct the Dirac operator and chirality operator on S^2 for spin $\frac{1}{2}$ and spin 1.

4 The Dirac and Chirality operators on S^2

We can construct a set of anti-commuting operators and call them the Dirac and chirality operators after checking that they have the right properties. Consider

$$D = (\Sigma_i - \gamma\Sigma_i\gamma)(\mathcal{L}_i + \Sigma_i) \tag{4.1}$$

where γ satisfies $\gamma^2 = 1$ and $\gamma\dagger = \gamma$. $\vec{\Sigma}$ is the spin j representation of SU(2). It is easy to check that this form of D in eq. (4.1) implies that

$$\{D, \gamma\} = 0 \tag{4.2}$$

as γ commutes with the total angular momentum $J_i = \mathcal{L}_i + \Sigma_i$. This follows from the following operator identity:

$$\{A, BC\} = \{A, B\}C - B[A, C] \tag{4.3}$$

Thus D and γ are Dirac and chirality operators.

D and γ for the Spin $\frac{1}{2}$ case. Let us now explicitly construct D and γ for the spin $\frac{1}{2}$ case.

In the fuzzy case $\vec{\sigma}.\vec{L}^L = L$ on the $L + \frac{1}{2}$ space and $\vec{\sigma}.\vec{L}^L = -(L + 1)$ on the $L - \frac{1}{2}$ space. Thus taking their continuum limits gives us $\vec{\sigma}.\hat{x} = \pm 1$ on these two spaces. An alternative way to find the eigenvalues of $\vec{\sigma}.\hat{x}$ without taking continuum limits of the fuzzy case is by noting that we can choose the direction of $\hat{x}$ to be along the third direction, which implies the eigenvalues of $\vec{\sigma}.\hat{x}$ are just the eigenvalues of σ_3 namely ± 1. This will be used extensively when we generalize to higher spins.

Using $\vec{\sigma}.\hat{x}$, we can construct the projectors onto the two spaces with $\vec{\sigma}.\hat{x} = \pm 1$:

$$P_1 = \frac{1 + \vec{\sigma}.\hat{x}}{2} \tag{4.4}$$

and

$$P_{-1} = \frac{1 - \vec{\sigma}.\hat{x}}{2} \tag{4.5}$$

Now for any projector P, $1 - 2P$ is an idempotent:

$$(1 - 2P)^2 = 1. \tag{4.6}$$

Thus from eq. (4.4) and eq. (4.5), we can read off the two chirality operators as $\pm\vec{\sigma}.\hat{x}$.

The Dirac operators corresponding to these two chirality operators are the same due to the form of the Dirac operator given by eq. (4.1).

We can compute D using the algebra of the Pauli matrices. That gives us

$$\sigma_i - (\vec{\sigma}.\hat{x})\sigma_i(\vec{\sigma}.\hat{x}) = \sigma_i - x_i(\vec{\sigma}.\hat{x}) \tag{4.7}$$

and thus from eq. (4.1),

$$D = \vec{\sigma}.\vec{\mathcal{L}} + \frac{1}{2} \tag{4.8}$$

which is the well-known continuum Dirac operator for spin $\frac{1}{2}$ on S^2 [11]

D and γ on S^2 for the spin 1 case. In a similar fashion we can find the chirality operators in the continuum for the spin 1 case by noting that the eigenvalues of $\vec{\Sigma}.\hat{x}$ are ± 1 and 0. We then write the projectors to the spaces where $\vec{\Sigma}.\hat{x}$ takes these three values and by writing these projectors as $\frac{1+\gamma}{2}$ we can read off the three chirality operators. They are

$$\gamma_1 = 1 - 2(\vec{\Sigma}.\hat{x})^2, \tag{4.9}$$
$$\gamma_2 = (\vec{\Sigma}.\hat{x})^2 + (\vec{\Sigma}.\hat{x}) - 1, \tag{4.10}$$
$$\gamma_3 = (\vec{\Sigma}.\hat{x})^2 - (\vec{\Sigma}.\hat{x}) - 1. \tag{4.11}$$

The Dirac operator corresponding to eq. (4.9) is found to be

$$D_1 = \vec{\Sigma}.\vec{\mathcal{L}} - (\vec{\Sigma}.\hat{x})^2 + 2. \tag{4.12}$$

The ones corresponding to the other chirality operators are unitarily equivalent to this one. The corresponding unitary operator transforms the eigenspace of $\vec{\Sigma}\hat{x}$ with eigenvalue 0 to either of the other eigenvalues. It is easy to write down the unitary operator connecting these chiralities if we take $\hat{x}$ to be in the third direction. If this is the case the three chiralities become

$$\gamma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.13}$$

$$\gamma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.14}$$

and

$$\gamma_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.15}$$

The unitary matrix transforming eq. (4.13) to eq. (4.14) is

$$U = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.16}$$

We do not know how the general unitary transform between the three chiralities will be. We suspect it to be an operator of the form e^{iD} where D is the Dirac operator.

Our fuzzy Dirac and chirality operators will have these as their continuum limits.

The Dirac operator in eq. (4.12) is found using the algebra of the spin 1 matrices [12] which is used to simplify

$$[\Sigma_i - (1-2(\vec{\Sigma}.\hat{x})^2)\Sigma_i(1-2(\vec{\Sigma}.\hat{x}))^2](\mathcal{L}_i+\Sigma_i). \tag{4.17}$$

We simplify the term in the square bracket after writing it in the form

$$[2\Sigma_i(\vec{\Sigma}.\hat{x})^2 + (\vec{\Sigma}.\hat{x})^2\Sigma_i - 4(\vec{\Sigma}.\hat{x})^2\Sigma_i(\vec{\Sigma}.\hat{x})^2]. \tag{4.18}$$

The first two terms in the above expression can be simplified using

$$\Sigma_i\Sigma_k\Sigma_j = \frac{i}{3}\varepsilon_{ikj} + \frac{1}{2}(\delta_{ik}\Sigma_j + \delta_{kj}\Sigma_i) + i\varepsilon_{ijm}Q_{km} \tag{4.19}$$

where Q_{km} is a symmetric tensor. This identity gives the sum of the first two terms as

$$A + B = 2\Sigma_i + 2(\vec{\Sigma}.\hat{x})x_i \tag{4.20}$$

where A and B are the first two terms in eq. (4.18). The identity in eq. (4.19) can also be used to simplify the third term in eq. (4.18) and we get

$$C = 2\Sigma_i(\vec{\Sigma}.\hat{x})^2 + 2x_i(\vec{\Sigma}.\hat{x}) - 4i\varepsilon_{ikm}Q_{jm}(\vec{\Sigma}.\hat{x})^2x_kx_j. \tag{4.21}$$

Using eq. (4.19), we can simplify this further to

$$C = 3x_i(\vec{\Sigma}.\hat{x}) + \Sigma_i + 2i\varepsilon_{ijm}Q_{km}x_kx_j - 4i\varepsilon_{ikm}Q_{jm}(\vec{\Sigma}.\hat{x})^2x_kx_j. \tag{4.22}$$

To evaluate this, we need to simplify the last term in the expression. That can be done using the following identities:

$$\Sigma_l\Sigma_n = \frac{2}{3}\delta_{ln} + \frac{i}{2}\varepsilon_{lno}\Sigma_o + Q_{ln} \tag{4.23}$$

and

$$\begin{aligned} Q_{jm}Q_{ln} = &\frac{1}{6}\left(\delta_{jl}\delta_{mn} + \delta_{jn}\delta_{lm} - \frac{2}{3}\delta_{jm}\delta_{ln}\right) \\ &- \frac{1}{4}\left(\delta_{jl}Q_{mn} + \delta_{jn}Q_{lm} + \delta_{mn}Q_{jl} + \delta_{ml}Q_{jn} - \frac{4}{3}\delta_{jm}Q_{ln} - \frac{4}{3}\delta_{ln}Q_{jm}\right) \\ &+ \frac{i}{8}(\delta_{jl}\varepsilon_{mnp}\Sigma_p + \delta_{jn}\varepsilon_{mlp}\Sigma_p + \delta_{ml}\varepsilon_{jnp}\Sigma_p + \delta_{mn}\varepsilon_{jlp}\Sigma_p). \end{aligned} \tag{4.24}$$

On using these two identities, the last term in eq. (4.22) becomes

$$2i\varepsilon_{ikm}Q_{lm}x_kx_l - x_i(\vec{\Sigma}.\hat{x}) + \Sigma_i \tag{4.25}$$

This can then be substituted in eq. (4.22) to get

$$C = 4x_i(\vec{\Sigma}.\hat{x}). \tag{4.26}$$

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With this, we obtain the following simple expression for $A + B - C$:

$$A + B - C = \Sigma_i - x_i(\vec{\Sigma}.\hat{x}) \tag{4.27}$$

Multiplying this with $(\vec{\mathcal{L}}_i + \vec{\Sigma}_i)$ gives the Dirac operator in eq. (4.12).

Next we write down the Dirac operators corresponding to the other chirality operators.

The Dirac operators corresponding to eq. (4.10) and eq. (4.11) are found to be

$$D_2 = (\vec{\Sigma}.\vec{\mathcal{L}} - (\vec{\Sigma}.\hat{x})^2 + 2) + 2(\vec{\Sigma}.\hat{x}) + \{\vec{\Sigma}.\vec{\mathcal{L}}, \vec{\Sigma}.\hat{x}\} \tag{4.28}$$

and

$$D_3 = (\vec{\Sigma}.\vec{\mathcal{L}} - (\vec{\Sigma}.\hat{x})^2 + 2) - 2(\vec{\Sigma}.\hat{x}) - \{\vec{\Sigma}.\vec{\mathcal{L}}, \vec{\Sigma}.\hat{x}\}. \tag{4.29}$$

These are found using the algebra of spin 1 matrices [12] as before.

These are the continuum limits which guide us in finding the fuzzy spin 1 Dirac operators. This will be explained in the next section where we discuss in detail the construction of the fuzzy spin 1 Dirac operator.

5 The fuzzy spin 1 Dirac operator

Consider $Mat(2L + 1) \bigotimes \mathbb{C}^3$, where $Mat(2L + 1)$ is the carrier space of spin $L \otimes L$ representation of SU(2) acting on left and right and $\mathbb{C}^3$ is the carrier space of the spin 1 representation of SU(2). When a spin L couples with spin 1, we have three possible spaces labeled by the values of the total angular momentum $L + 1, L$ and $L - 1$. So we have six projectors and as in eq. (3.3) and eq. (3.4) we can construct the corresponding generators of the GW algebra. Thus we have a table similar to the one in eq. (3.3) and eq. (3.4):

$$P^{L,R}_{L+1} : \quad \Gamma^L_{L+1} \quad \Gamma^R_{L+1} \tag{5.1}$$

$$P^{L,R}_{L} : \quad \Gamma^L_{L} \quad \Gamma^R_{L} \tag{5.2}$$

$$P^{L,R}_{L-1} : \quad \Gamma^L_{L-1} \quad \Gamma^R_{L-1} \tag{5.3}$$

The notation used here is similar to the one used in section 3.

The three projectors corresponding to the left angular momentum coupling to spin 1 are

$$P^L_{L+1} = \frac{(\vec{\Sigma}.\vec{L}^L + L + 1)(\vec{\Sigma}.\vec{L}^L + 1)}{(L+1)(2L+1)} \tag{5.4}$$

$$P^L_{L} = -\frac{(\vec{\Sigma}.\vec{L}^L - L)(\vec{\Sigma}.\vec{L}^L + L + 1)}{L(L+1)} \tag{5.5}$$

$$P^L_{L-1} = \frac{(\vec{\Sigma}.\vec{L}^L - L)(\vec{\Sigma}.\vec{L}^L + 1)}{(2L+1)L} \tag{5.6}$$

while the corresponding right projectors are obtained from above by substituting $\vec{L}^L$ by $-\vec{L}^R$.

Writing each projector as $\frac{1+\Gamma}{2}$ and $\vec{L}$ as $\vec{L}^L$ or $-\vec{L}^R$, we can find the generators of the GW algebra for each of the projectors above. Let us write down the relevant generators whose combinations give the fuzzy Dirac and chirality operators having the right continuum limits which we found in the previous section.

$$\Gamma^L_{L+1} = \frac{2(\vec{\Sigma}.\vec{L}^L + L + 1)(\vec{\Sigma}.\vec{L}^L + 1) - (L+1)(2L+1)}{(L+1)(2L+1)} \tag{5.7}$$

$$\Gamma^R_{L+1} = \frac{2(-\vec{\Sigma}.\vec{L}^R + L + 1)(-\vec{\Sigma}.\vec{L}^R + 1) - (L+1)(2L+1)}{(L+1)(2L+1)} \tag{5.8}$$

$$\Gamma^L_{L-1} = \frac{2(\vec{\Sigma}.\vec{L}^L - L)(\vec{\Sigma}.\vec{L}^L + 1) - L(2L+1)}{L(2L+1)} \tag{5.9}$$

$$\Gamma^R_{L-1} = \frac{2(\vec{\Sigma}.\vec{L}^R + L)(\vec{\Sigma}.\vec{L}^R - 1) - L(2L+1)}{L(2L+1)} \tag{5.10}$$

We can immediately see that $\frac{\Gamma^L_{L-1} \pm \Gamma^R_{L+1}}{2}$, are chirality and Dirac operators (the latter upto an overall constant) for the fuzzy sphere by checking their continuum limits. Thus as $L \to \infty$,

$$\frac{\Gamma^L_{L-1} + \Gamma^R_{L+1}}{2} \to (\vec{\Sigma}.\hat{x})^2 - \vec{\Sigma}.\hat{x} - 1, \tag{5.11}$$

which is a chirality operator for the spin 1 case in the continuum which we encountered in the previous section. Also

$$\lim_{L\to\infty} L\left(\frac{\Gamma^L_{L-1} - \Gamma^R_{L+1}}{2}\right) = -(\vec{\Sigma}.\vec{\mathcal{L}} - (\vec{\Sigma}.\hat{x})^2 + 2) \tag{5.12}$$

is the corresponding Dirac operator as $L(\frac{\Gamma^L_{L-1} - \Gamma^R_{L+1}}{2})$ anti-commutes with $\frac{\Gamma^L_{L-1} + \Gamma^R_{L+1}}{2}$. The Dirac operator got from the fuzzy case in eq. (5.12)is unitarily equivalent to the one got in eq. (4.28). This can be seen as a consequence of the fact that the chiralities corresponding to these Dirac operators are unitarily equivalent. Eq. (5.12) can be seen by substituting the expressions for Γ^L_{L-1} and Γ^R_{L+1} from eq. (5.9) and eq. (5.8) respectively and grouping terms similar in the order of $\vec{L}^L$ and $\vec{L}^R$.

Here we note the order L term in the expression

$$L\frac{((\vec{\Sigma}.\vec{L}^L)^2 - (\vec{\Sigma}.\vec{L}^R)^2)}{(L+1)(2L+1)} \tag{5.13}$$

got by grouping the second order terms. As $L \to \infty$ this term goes to $-\frac{(\vec{\Sigma}.\hat{x})}{2}$. This can be understood easily by noting that $[L^L_i, L^L_j] = i\varepsilon_{ijk}L^L_k$, produces first order terms in L and these commutators arise when we expand $L^L_i L^L_j$ as a sum of a commutator and an anticommutator. However, this is just the highest order term and this limit is not exact. We shall in fact see later that the exact limit is different from this involving a first order differential term thereby changing the form of the Dirac operator. But we will show it to be unitarily equivalent to the above Dirac operator.

Similarly we find the chirality and Dirac operators $\frac{\Gamma^L_{L+1} + \Gamma^R_{L-1}}{2}$ for the fuzzy sphere(the latter upto a constant) and their continuum limits.

$$\frac{\Gamma^L_{L+1} + \Gamma^R_{L-1}}{2} \to (\vec{\Sigma}.\hat{x})^2 + \vec{\Sigma}.\hat{x} - 1 \tag{5.14}$$

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and

$$L\left(\frac{\Gamma^L_{L+1}-\Gamma^R_{L-1}}{2}\right) \rightarrow (\vec{\Sigma}.\vec{\mathcal{L}} - (\vec{\Sigma}.\hat{x})^2 + 2) \tag{5.15}$$

as $L \rightarrow \infty$. The Dirac operator got from the fuzzy case in eq. (5.12)is unitarily equivalent to the one got in eq. (4.29). Again this can be seen as a consequence of the fact that the chiralities corresponding to these Dirac operators are unitarily equivalent. We will remark more about this later.

We can also see that γ_1 in eq. (4.9) is got by taking the continuum limit of

$$\frac{\Gamma^L_L + \Gamma^R_L}{2} \tag{5.16}$$

where

$$\Gamma^L_L = \frac{-2(\vec{\Sigma}.\vec{L}^L - L)(\vec{\Sigma}.\vec{L}^L + L + 1) - L(L+1)}{L(L+1)} \tag{5.17}$$

$$\Gamma^R_L = \frac{2(\vec{\Sigma}.\vec{L}^R + L)(-\vec{\Sigma}.\vec{L}^R + L + 1) - L(L+1)}{L(L+1)} \tag{5.18}$$

This implies $L(\frac{\Gamma^L_L - \Gamma^R_L}{2})$ goes to the corresponding Dirac operator. Thus $\frac{\Gamma^L_L + \Gamma^R_L}{2}$ and constant times $\frac{\Gamma^L_L - \Gamma^R_L}{2}$ can also serve as chirality and Dirac operators.

The continuum limit of the combination $\Gamma^R_L + \Gamma^L_{L+1}$ goes to $\vec{\Sigma}.\hat{x} - (\vec{\Sigma}.\hat{x})^2$ which is not part of the chiralities we obtained in the continuum in section 4. They are not unitarily to equivalent to any of those obtained in section 4 either. Other combinations like $\Gamma^R_L + \Gamma^L_{L-1}$ go to a chirality we do not have in the continuum as formulated in section 4. The combinations anticommuting with these namely $L(\Gamma^R_L - \Gamma^L_{L+1})$ and $L(\Gamma^R_L + \Gamma^L_{L-1})$ do not have proper continuum limits, in fact they diverge. Hence we discard these combinations.

6 Generalizing to higher spins

The projectors to spaces, got by coupling L to higher spins contain more factors increasing the order in $\vec{L}^{L,R}$ and making the expressions look complicated. We observe the kind of terms that can emerge from simplifying these expressions and formulate rules to take their continuum limits.

We first carefully look at the spin $\frac{3}{2}$ case and use this to generalize to terms emerging from higher spins. We have eight projectors in this case which are $P^{L,R}_{L+\frac{3}{2}}$, $P^{L,R}_{L+\frac{1}{2}}$, $P^{L,R}_{L-\frac{1}{2}}$, $P^{L,R}_{L-\frac{3}{2}}$. We can construct the generators of the GW algebra from each of these projectors and thus construct a table similar to that shown in eqs. (5.1)–(5.3). From this table, let us take the relevant Γ operators whose combination gives us the fuzzy Dirac operator. Consider

$$\Gamma^L_{L+\frac{3}{2}} = \frac{(2\vec{\Sigma}.\vec{L}^L - L + 3)(2\vec{\Sigma}.\vec{L}^L + L + 4)(2\vec{\Sigma}.\vec{L}^L + 3L + 3) - 6(L+1)(2L+3)(2L+1)}{6(L+1)(2L+1)(2L+3)} \tag{6.1}$$

$$\Gamma^R_{L-\frac{3}{2}} = \frac{(-2\vec{\Sigma}.\vec{L}^R - L + 3)(-2\vec{\Sigma}.\vec{L}^R + L + 4)(2\vec{\Sigma}.\vec{L}^R + 3L) - 6L(2L-1)(2L+1)}{6L(2L+1)(2L-1)} \tag{6.2}$$

Now as $L \to \infty$,

$$\Gamma^L_{L+\frac{3}{2}} + \Gamma^R_{L-\frac{3}{2}} \to \frac{8(\vec{\Sigma}.\hat{x})^3 - 2\vec{\Sigma}.\hat{x} + 12(\vec{\Sigma}.\hat{x})^2 - 27}{24} \tag{6.3}$$

The Dirac operator corresponding to this can be got from taking the continuum limit of $L(\Gamma^L_{L+\frac{3}{2}} - \Gamma^R_{L-\frac{3}{2}})$. We will look at the possible terms we will be coming across in the process of taking the limits of the Dirac operators. In the case of spin $\frac{3}{2}$, we see the following term:

$$L\left(\frac{(\vec{\Sigma}.\vec{L}^L)^3 - (\vec{\Sigma}.\vec{L}^R)^3}{L^3}\right) \tag{6.4}$$

There is also a constant factor multiplying this. However this is not important for us right now as we are formulating rules for taking continuum limits of such terms.

Let us see how to take this continuum limit. For this consider

$$\frac{(\vec{\Sigma}.\vec{L}^L)^3}{L^2} = \frac{1}{L^2}(\vec{\Sigma}.\vec{\mathcal{L}} + \vec{\Sigma}.\vec{L}^R)^3 \tag{6.5}$$

$$= \frac{1}{L^2}\Big[(\vec{\Sigma}.\vec{\mathcal{L}})^3 + (\vec{\Sigma}.\vec{L}^R)^2(\vec{\Sigma}.\vec{\mathcal{L}}) + \{\vec{\Sigma}.\vec{\mathcal{L}}, \vec{\Sigma}.\vec{L}^R\}\vec{\Sigma}.\vec{\mathcal{L}} + (\vec{\Sigma}.\vec{\mathcal{L}})^2\vec{\Sigma}.\vec{L}^R + (\vec{\Sigma}.\vec{L}^R)^3 + \{\vec{\Sigma}.\vec{\mathcal{L}}, \vec{\Sigma}.\vec{L}^R\}\vec{\Sigma}.\vec{L}^R\Big]. \tag{6.6}$$

Here we have written $\vec{L}^L = \vec{\mathcal{L}} + \vec{L}^R$ where $\vec{\mathcal{L}}$ is the first order differential operator in the continuum. In the previous equation we note that the $(\vec{\Sigma}.\vec{L}^R)^3$ term cancels the $-(\vec{\Sigma}.\vec{L}^R)^3$ in equation eq. (6.4). When $L \to \infty$, the order 1 terms in $\vec{L}^R$ go away. The $(\vec{\Sigma}.\vec{\mathcal{L}})^3$ also goes away in the continuum as we take the limit. So we are left with the following terms that have a non-zero limit

$$\frac{(\vec{\Sigma}.\vec{L}^L)^3}{L^2} = \frac{1}{L^2}\Big[\{\vec{\Sigma}.\vec{\mathcal{L}}, (\vec{\Sigma}.\vec{L}^R)^2\} + (\vec{\Sigma}.\vec{L}^R)(\vec{\Sigma}.\vec{\mathcal{L}})(\vec{\Sigma}.\vec{L}^R)\Big]. \tag{6.7}$$

This is the following self-adjoint operator in the continuum:

$$\{\vec{\Sigma}.\vec{\mathcal{L}}, (\vec{\Sigma}.\hat{x})^2\} + (\vec{\Sigma}.\hat{x})(\vec{\Sigma}.\vec{\mathcal{L}})(\vec{\Sigma}.\hat{x}). \tag{6.8}$$

The other terms we find in the expression for the fuzzy Dirac operator for the spin $\frac{3}{2}$ case involve powers of $\vec{L}^L$ and $\vec{L}^R$ less than 3 and their continuum limits were already found while we evaluated the corresponding continuum limits in the spin 1 and the spin $\frac{1}{2}$ case.

At this point we make a crucial observation that the limits we are taking are all independent of the algebra of the spin matrices $\vec{\Sigma}$. This is the reason why we need not bother about the order 1 and 2 terms in the spin $\frac{3}{2}$ case, though the spin matrices $\vec{\Sigma}$ are different from those in the spin 1 case.

We are interested in finding the limits of expressions of the form eq. (6.4), which are similar in the case of all spins, but with higher powers of $\vec{L}^L$ and $\vec{L}^R$.

Consider first

$$\frac{(\vec{\Sigma}.\vec{L}^L)^4 - (\vec{\Sigma}.\vec{L}^R)^4}{L^3} = \frac{1}{L^3}\left((\vec{\Sigma}.(\vec{\mathcal{L}} + \vec{L}^R))^4 - (\vec{\Sigma}.\vec{L}^R)^4\right) \tag{6.9}$$

$$= \frac{1}{L^3}\left([(\vec{\Sigma}.\vec{\mathcal{L}})^2 + (\vec{\Sigma}.\vec{L}^R)^2 + \{\vec{\Sigma}.\vec{L}^R, \vec{\Sigma}.\vec{\mathcal{L}}\}][(\vec{\Sigma}.\vec{\mathcal{L}})^2 + (\vec{\Sigma}.\vec{L}^R)^2 + \{\vec{\Sigma}.\vec{L}^R, \vec{\Sigma}.\vec{\mathcal{L}}\}] - (\vec{\Sigma}.\vec{L}^R)^4\right) \tag{6.10}$$

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In the above expression, only the order 3 terms in L^R have a non-zero continuum limit. The $(\vec{\Sigma}.\vec{L}^R)^4$ term cancels just as it did in expression eq. (6.4). The terms with non-zero limit are

$$\frac{1}{L^3}[\{\vec{\Sigma}.\vec{L}^R, \vec{\Sigma}.\vec{\mathcal{L}}\}(\vec{\Sigma}.\vec{L}^R)^2 + (\vec{\Sigma}.\vec{L}^R)^2\{\vec{\Sigma}.\vec{L}^R, \vec{\Sigma}.\vec{\mathcal{L}}\}] \tag{6.11}$$

As $L \to \infty$ this term goes to the following non zero, self-adjoint expression

$$\{\vec{\Sigma}.\vec{\mathcal{L}}, (\vec{\Sigma}.\hat{x})^3\} + \{\vec{\Sigma}.\vec{\mathcal{L}}, \vec{\Sigma}.\hat{x}(\vec{\Sigma}.\vec{\mathcal{L}})\vec{\Sigma}\hat{x}\} \tag{6.12}$$

Looking at this pattern and using the fact that we are just applying the binomial expansion in this computation, we can write a general rule for computing the continuum limit for order n terms. For this we consider

$$\frac{1}{L^{n-1}}[(\vec{\Sigma}.\vec{L}^L)^n - (\vec{\Sigma}.\vec{L}^R)^n] \tag{6.13}$$

Again we write $\vec{L}^L = \vec{\mathcal{L}} + \vec{L}^R$ and expand $(\vec{\Sigma}.\vec{L}^L)^n$ using the binomial expansion. As in previous cases the $(\vec{\Sigma}.\vec{L}^R)^n$ term gets canceled and we need to pick only the order $n-1$ terms in $\vec{L}^R$ as these are the only terms having a non-zero continuum limit. Since the continuum operator has to be self-adjoint and the terms occurring in the expansion are all those occurring in a binomial expansion, it is easy to see that the terms having a non-zero limit can be given as the following sum:

$$\frac{1}{L^{n-1}}\left(\sum_{k=0}^{n-1}(\vec{\Sigma}.\vec{L}^R)^{n-1-k}(\vec{\Sigma}.\vec{\mathcal{L}})(\vec{\Sigma}.\vec{L}^R)^k\right) \tag{6.14}$$

It is clear from this expression that we only have terms of order $n-1$ in $\vec{L}^R$ here and we immediately see the continuum limit of this expression as

$$\sum_{k=0}^{n-1}(\vec{\Sigma}.\hat{x})^{n-1-k}(\vec{\Sigma}.\vec{\mathcal{L}})(\vec{\Sigma}.\hat{x})^k. \tag{6.15}$$

Thus when considering the expression for the Dirac operator for any spin j, the highest order term in $\vec{\Sigma}.\vec{L}^L$ has a power $n = 2j$ and other terms decrease from $2j$ to 1. We have just seen how to take the continuum limit of each of these terms with our general rules. We also encounter polynomials in L in these expressions whose limits are easy to take. Apart from grouping terms of similar order as in eq. (6.4), we will also encounter $\vec{\Sigma}.\vec{L}^{L,R}$ of various orders which cannot be grouped as in eq. (6.4). The process of taking limits for such terms is straightforward and we will not elaborate them here.

Verifying for spin $\frac{1}{2}$. We see that the highest order term is $n = 1$ and so the Dirac operator in the continuum consists of just $\vec{\Sigma}.\vec{\mathcal{L}}$. Then we have a polynomial in $\vec{L}^{L,R}$ in the next order whose limit combined with the limit of the first order term gives $\vec{\Sigma}.\vec{\mathcal{L}} + 1$ as before, where $\vec{\Sigma} = \frac{\vec{\sigma}}{2}$.

Verifying for spin 1. For spin 1, we have $n = 2$ as the highest order term and this gives the term $\{\vec{\Sigma}.\hat{x}, \vec{\Sigma}.\vec{\mathcal{L}}\}$ according to our general rule. We should then look at the $n = 1$ term which gives a term proportional to $\vec{\Sigma}.\vec{\mathcal{L}}$. By taking the continuum limit of $L(\frac{\Gamma^L_{L-1}-\Gamma^R_{L+1}}{2})$ according to the rules from the previous section, we get

$$D' = \frac{\{\vec{\Sigma}.\vec{\mathcal{L}}, \vec{\Sigma}.\hat{x}\}}{2} - \left(\frac{\vec{\Sigma}.\vec{\mathcal{L}}}{2} - \frac{(\vec{\Sigma}.\hat{x})^2}{2} + 1\right) + \vec{\Sigma}.\hat{x}. \tag{6.16}$$

This operator is $-\frac{1}{2}$ times the Dirac operator got in eq. (4.29). The constant factor of $-\frac{1}{2}$ can be absorbed in the scale factor multiplying the fuzzy Dirac operator. In a similar way the other continuum Dirac operators can be got by taking the limits of the correct fuzzy versions.

This verifies the rules we formulated for the know cases of spin 1 and spin $\frac{1}{2}$.

Showing unitary equivalences. We speculate that D' is unitarily equivalent to the Dirac operator D we got in eq. (4.12). We are not however able to exactly prove it. The basis of our speculation is the unitary equivalence of the Dirac operators of [3] and [10] proved in [9]. Following that approach we consider the following unitary transformation by the unitary operator generated by the chirality operator γ:

$$D' = \{\exp i\theta\gamma\}\, D\, \{\exp -i\theta\gamma\} \tag{6.17}$$

It follows from $\gamma^2 = 1$ and $\{D, \gamma\} = 0$, that the previous equation can be written as

$$D' = \{\exp 2i\theta\gamma\}\, D \tag{6.18}$$

which is

$$D' = \cos 2\theta D + i \sin 2\theta\gamma D. \tag{6.19}$$

Substituting for γ from eq. (4.10) or eq. (4.11) we calculate the second term in eq. (6.19) to check the equivalence. We get

$$\gamma D = \left[(\vec{\Sigma}.\hat{x})^2 - \vec{\Sigma}.\hat{x}\right] \vec{\Sigma}.\vec{\mathcal{L}} + (\vec{\Sigma}.\hat{x})^2 - \vec{\Sigma}.\hat{x} - D. \tag{6.20}$$

Most of the terms in eq. (6.16) are seen in the above expression except $(\vec{\Sigma}.\hat{x})^2\vec{\Sigma}.\vec{\mathcal{L}}$. This term can be simplified using eq. (4.19) to get:

$$(\vec{\Sigma}.\hat{x})^2\vec{\Sigma}.\vec{\mathcal{L}} = \frac{1}{2}\vec{\Sigma}.\vec{\mathcal{L}} + i\epsilon_{ijm}Q_{km}\hat{x}_k\hat{x}_i\mathcal{L}_j. \tag{6.21}$$

Unitary equivalence will consist in picking θ so that eq. (6.19) becomes eq. (6.16). Unfortunately we see terms of the form $(\vec{\Sigma}.\hat{x})^2\vec{\Sigma}.\vec{\mathcal{L}}$ in eq. (6.19) which are not present in eq. (6.16). Perhaps we must make an additional unitary transformation with a unitary operator commuting with γ. We do not know what such an operator can be.

We can construct more Dirac operators in the continuum starting from the one given in eq. (4.1). We do this by first observing that all choices of γ in the continuum depend only on $\vec{\Sigma}.\hat{x}$. Hence if P is a function of a variable η with a convergent power series expansion in η, then

$$D^P = \{P(\vec{\Sigma}.\hat{x})(\Sigma_i - \gamma\Sigma_i\gamma) + (\Sigma_i - \gamma\Sigma_i\gamma)P(\vec{\Sigma}.\hat{x})^\dagger\}(\mathcal{L}_i + \Sigma_i) \tag{6.22}$$

is also self-adjoint, anti-commutes with γ and is hence also a Dirac operator.

It is not clear if different choices of P lead to unitarily equivalent Dirac operators (after an overall scaling) or not. A definitive answer to such questions can be obtained by calculating the spectrum of these operators. Since we are not able to do so analytically, we are now doing so numerically [13].

7 Summary of rules for finding the fuzzy Dirac operator

Half-integral spins. In this case, we have an even number of projectors and hence an even number of chiralities in the continuum. We can easily find all the chiralities in the continuum as they are just got from constructing projectors to various spaces labeled by the eigenvalues of $\vec{\Sigma}.\hat{x}$.

Next we list the projectors in the fuzzy case and construct the corresponding GW systems for each of them. So we have tables similar to the ones in eqs. (5.1)–(5.3). Then we consider the construction of the correct combination of the generators of the various GW systems, which go to the chiralities found in the continuum previously, as we take the continuum limit.

The claim is: The chiralities got from the projectors to the spaces labeled by j and $-j$ in the continuum are got by taking the continuum limits of

$$\frac{\Gamma^L_{L+j} + \Gamma^R_{L-j}}{2} \tag{7.1}$$

and

$$\frac{\Gamma^L_{L-j} + \Gamma^R_{L+j}}{2} \tag{7.2}$$

respectively.

We now prove this claim:

Consider spin j coupling to the orbital part l. Then if we project to the $l+j-k$ space, it is easy to see that

$$\textit{Spectrum of}\ \ \vec{\Sigma}.\vec{L}^L \in lj + \frac{k}{2}[k-1-2l-2j] \tag{7.3}$$

where $k = 0, 1, ..., 2j$. We use this spectrum to construct the projectors to the above spaces.

It then follows from definition that

$$\frac{\Gamma^L_{l+j} + \Gamma^R_{l-j}}{2} = P^L_{l+j} + P^R_{l-j} - 1 \tag{7.4}$$

where $P^{L,R}$ denotes the left or right projector to the corresponding space, indicated in the suffix. Taking the continuum limit, we get

$$\lim_{l\to\infty} P^L_{l+j} + P^R_{l-j} - 1 = \prod_{k=1}^{2j} \frac{(\vec{\Sigma}.\hat{x} - j + k)}{k} + \prod_{k=0}^{2j-1} \frac{(-\vec{\Sigma}.\hat{x} - j + k)}{(-2j+k)} - 1. \tag{7.5}$$

Pulling out the minus signs in the second expression we get

$$\lim_{l\to\infty} P^L_{l+j} + P^R_{l-j} - 1 = \frac{\prod_{k=1}^{2j}(\vec{\Sigma}.\hat{x} - j + k)}{(2j)!} + (-1)^{4j}\frac{\prod_{k=0}^{2j-1}(\vec{\Sigma}.\hat{x} + j - k)}{(2j)!} - 1. \qquad (7.6)$$

Since $4j$ is even for both integral and half-integral j, observing that $\prod_{k=1}^{2j}\vec{\Sigma}.\hat{x} - (j-k) = \prod_{k=0}^{2j-1}\vec{\Sigma}.\hat{x} + (j-k)$, we get

$$\lim_{l\to\infty} \frac{\Gamma^L_{l+j} + \Gamma^R_{l-j}}{2} = 2\frac{\prod_{k=1}^{2j}(\vec{\Sigma}.\hat{x} - j + k)}{(2j)!} - 1. \qquad (7.7)$$

This is exactly the expression for the chirality operator got in the continuum from the projector to the space where $\vec{\Sigma}.\hat{x} = j$.

Now since, $L\left(\frac{\Gamma^L_{L+j} - \Gamma^R_{L-j}}{2}\right)$ and $\frac{\Gamma^L_{L+j} + \Gamma^R_{L-j}}{2}$ anticommute in the fuzzy case, they will continue to do so as we take the continuum limit. So we can be sure that

$$L\left(\frac{\Gamma^L_{L+j} - \Gamma^R_{L-j}}{2}\right) \qquad (7.8)$$

gives us the fuzzy Dirac operator corresponding to this chirality.

We can follow the same procedure to get the remaining fuzzy Dirac and chirality operators, exhausting all possibilities.

Integral spins. In this case, we have an odd number of projectors and hence an odd number of chiralities in the continuum. We then proceed as we did for the case of half-integral spins and we note that all the arguments go through, except when it comes to the Dirac operator corresponding to the chirality obtained from the projector to the space where $\vec{\Sigma}.\hat{x} = 0$. In this case, we construct the fuzzy analogues from the generators of the GW system obtained from the left and right projectors to the $L+0$ space alone. We cannot mix the generators of the GW system got from this projector with the generators obtained from the projectors to other spaces as we get diverging continuum limits. We omit the simple details for showing this result.

8 Index theory for the spin j Dirac operator

The index of the Dirac operator can be computed by counting the number of zero modes. These zero modes are eigenstates of the Dirac operator spanning a subspace left invariant by the chirality operator. Thus if chirality is diagonalised in this subspace of zero modes and the dimensions of the zero mode subspaces with $\gamma = \pm 1$ are $n_{L,R}$, the index of the Dirac operator is $n_L - n_R$. There will be a minimum of $n_L - n_R$ linearly independent zero modes of the Dirac operator with $\gamma = 1 (\gamma = -1)$, if $n_L \geq n_R$ ($n_L \leq n_R$), respectively.

We can compute the index as follows [1, 14]. Consider the instanton sectors of S^2, which correspond to U(1) bundles thereon. On S^2_F, projective modules substitute for sections of bundles.

We build the projective modules on S^2_F by introducing a spin T representation of SU(2) whose carrier space is $\mathbb{C}^{2T+1}$. We then consider, $Mat(2L+1) \otimes \mathbb{C}^{2T+1}$, on which SU(2)

$\vec{L}^L + \vec{T} - \vec{L}^R$	$\vec{\Sigma}$	$\vec{J}$
$0 + T \rightarrow$	$-\frac{1}{2} \rightarrow$	$T - \frac{1}{2}$
$0 + T \rightarrow$	$\frac{1}{2} \rightarrow$	$T + \frac{1}{2}$
$1 + T \rightarrow$	$-\frac{1}{2} \rightarrow$	$T + \frac{1}{2}$
$1 + T \rightarrow$	$\frac{1}{2} \rightarrow$	$T + \frac{3}{2}$
$2 + T \rightarrow$	$-\frac{1}{2} \rightarrow$	$T + \frac{3}{2}$
$\vdots$	$\vdots$	$\vdots$
$2L + T - 1 \rightarrow$	$\frac{1}{2} \rightarrow$	$2L + T - \frac{1}{2}$
$2L + T \rightarrow$	$-\frac{1}{2} \rightarrow$	$2L + T - \frac{1}{2}$
$2L + T \rightarrow$	$+\frac{1}{2} \rightarrow$	$2L + T + \frac{1}{2}$

Table 1. Method to find unpaired eigenstates.

acts with generators $\vec{L}^L + \vec{T}$. Then we consider, $Mat(2L+1) \otimes \mathbb{C}^{2T+1} \otimes \mathbb{C}^{2j+1}$, the space where the fuzzy spin j Dirac operator with instanton coupling acts. The desired projective modules are then constructed by considering $P^{L\pm T} Mat(2L+1) \otimes \mathbb{C}^{2T+1} \otimes \mathbb{C}^{2j+1}$, where $P^{L\pm T}$ is the projector to the space where $\vec{L}^L + \vec{T}$ couple to $L+T$ and $L-T$ respectively. The different projectors obtaining by varying T as well correspond to different Chern numbers which classify the projective modules in the continuum and in the fuzzy case. Using these projectors we can construct their corresponding GW systems and hence the fuzzy Dirac operators with instanton coupling. We do not explicitly show the construction of the projective modules for a general spin j here. For details regarding spin $\frac{1}{2}$, see [1].

Next we find the unpaired eigenstates obtained by combining the four angular momenta, namely $\vec{L}^L, \vec{T}, -\vec{L}^R, \vec{\Sigma}$, to get the total angular momentum $\vec{J}$. Unpaired eigenstates are those whose eigenvalues are got by combining the four angular momenta in a unique way. These are eigenstates of the total angular momentum $\vec{J}$. These are also eigenstates of the Dirac operator as $\vec{J}$ commutes with the Dirac operator. The method of counting the number of unpaired eigenstates is illustrated in table 1, where we have considered the case where $\vec{\Sigma} = \frac{1}{2}$. We note that the states with total angular momentum $T - \frac{1}{2}$ and $2L+T+\frac{1}{2}$ are the unpaired ones as they occur just once in table 1. The latter is the top mode and we can discard it as it does not agree with the values obtained in the continuum [15], (see page 95, chapter 8 of [1]). We are then left with the space whose value of total angular momentum is $T - \frac{1}{2}$ and the dimension of this space is $2T$. This is the number of zero modes of the Dirac operator and hence its index. This space is left invariant by the chirality operator.

This procedure can be carried out for any spin j. When we do this, we find that the only unpaired eigenstate, discarding the top mode, is the one with the eigenvalue $T - j$. This is also the minimum value of the total angular momentum. This gives us $2(T-j)+1$ as the number of zero modes and this is the index of these Dirac operators.

We can verify for the familiar [1] spin $\frac{1}{2}$ case that this gives $2T$. For the case of spin 1, this gives $2T - 1$.

9 Conclusions

We have seen that we can construct Dirac operators for any spin on the fuzzy 2-sphere. We made use of the properties of the projectors to various spaces to achieve this construction and by formulating rules to take their continuum limits we found these operators on the commutative 2-sphere as well. A general construction of the Dirac and chirality operators on the continuum 2-sphere was shown.

Formulating the gauge sectors of these operators in the fuzzy case [1] and taking their continuum limits, we can also get equations with interactions on S^2 and S^2_F.

We can construct the Dirac and chirality operators on $\mathbb{R}^2$. We did not show this construction here as it is quite straight forward and can be done using our general methods for constructing them. Moreover we did not obtain any new result by considering them.

We are examining the spectrum of these Dirac operators on S^2 and S^2_F. We could not find them analytically and so we are trying to do it numerically [13]. We are also studying quantum field theories associated with these operators by functional integral techniques.

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